11. Kurzhanskii, A. B., Differential convergence games with restricted phase coordinates. Dokl. Akad. Nauk SSSR Vol. 192, №3, 1970.
12. Gusiatnikov, P. B. and Nikol'skii, M. S., On the optimality of the pursuit time. Dokl. Akad. Nauk SSSR Vol. 184, №3, 1959.
13. Filippov, A. F., On certain problems of optimal control theory. Vestnik Moskov. Gos. Univ., Ser. Mat. Mekh., №2, 1959.
14. Banks, H. T., Representations for solutions of linear functional equations. Techn. Report, 2, 1968.
15. Halanay, A. Differential Equations. New York-London, Acad. Press, 1966.

Translated by A. Y.

OPTIMIZATION OF THE TRACKING PROCESS WITH INFORMATION LAG

PMM Vol. 35, №2, 1971, pp. 312-320
V. B. KOLMANOVSKII
(Moscow)
(Received March 9, 1970)

Problems of optimal control with incomplete information are of considerable interest in connection with practical control problems. In the present paper we investigate the problem of optimizing the process of tracking an object in the case of incomplete and inexact data on its position. The errors in the measured data are due to: (1) information lag, (2) the presence of random disturbances in the measuring instruments. Certain assumptions enable us to educe the problem of determining the optimal tracking law to an ordinary optimal control problem. The optimal tracking law is obtained in explicit form for certain quality criteria.

1. Let the motion of the object under investigation be described by the system of differential equations

$$x^{\cdot}(t) = A(t)x(t) + f(t) \tag{1.1}$$

and let the tracking system have available to it the vector $y(t)$ given by

$$y(t) = \int_0^t Q(s)\, x(s-h)\, ds + \int_0^t \sigma(s)\, d\xi(s) \tag{1.2}$$

where the l-dimensional vectors $x(t)$ and $y(t)$ belong the the Euclidean space E_l.

Unless otherwise indicated, the vectors from E_l occurring below are to be understood as column vectors; The jth coordinate of a vector will be denoted by the same letter as the vector with the subscript j. For example, the vector $x(t) = (x_1(t), \ldots, x_l(t))'$; here and below primes indicate transposition.

We assume that the following restrictions on the coefficients of Eqs. (1.1), (1.2) are fulfilled throughout: the $(l \times l)$-dimensional matrices $A(t)$, $Q(t)$, $\sigma(t)$ and the vector $f(t) \in E_l$; the elements of $f(t)$ and $A(t)$ are continuous, and the elements of $\sigma(t)$ and $Q(t)$ are Borel-measurable and bounded; the constant $h \geqslant 0$; finally, $\xi(t)$ which is a Wiener random process, assumes values from E_l and has independent components normalized by the conditions $\xi(0) = 0$, $M\xi_i(t) = 0$, $M\xi_i^2(t) = t$, $i = 1, \ldots, l$ (the letter M denotes the mathematical expectation); the matrix $B(t) = \sigma(t)\sigma'(t)$ is nondegenerate. Here Eq. (1.2) is to be undestood as an Ito stochastic differential equation.

The tracking $y(t)$ is conducted during the interval $0 \leqslant t \leqslant T$, and the initial distribution of the vector $x(0)$ is given. The distribution is normal with a known mathematical expectation $m_0 \in E_l$ and a nondegenerate correlation martix D_0; it does not depend on the distributions of the process $\xi(t)$. This and the properties of the coefficients of Eqs. (1.1) and (1.2) imply that the conditional distribution $x(t)$ is also normal under the condition $y(s)$, $0 \leqslant s \leqslant t$.

The possibility of varying the matrix $Q(t)$ (which defines the content of the measurements) and the matrix $B(t)$ (which defines the accuracy of the measurements) during tracking gives rise to various problems of optimal selection of these matrices. Some of these problems are considered below.

We note that closely related questions in the case $h = 0$ are considered in [1–3], where the choice of the matrices $Q(t)$ and $B(t)$ which are in some sense optimal, is solved with the aid of the Pontriagin maximum principle. In [1] the maximum principle is used to reduce the problem to a system of transcendental equations.

The maximum principle can also be used to solve the problems considered in the present paper. However, since the random quantity $x(0)$ is Gaussian, and since Eqs. (1.1), (1.2) are linear, the initial problem on the choice of the matrices $B(t)$ and $Q(t)$ can also be reduced to the associated problem of moments; another way of dealing with the problem is by the classical methods of the calculus of variations. The latter methods make it possible to find the optimal tracking mode in explicit form in certain situations.

2. We say that there is no tracking at the instant $s \in [0, T]$ if for this s all the elements of the matrix $Q(s)$ are equal to zero. By $m(t)$ and $D(t)$ we denote, respectively, the conditional mathematical expectation and conditional dispersion matrix of the process $x(t)$ in the presence of observations $y(t)$, and the unconditional mathematical expectation and unconditional dispersion matrix in the absence of observations. Here $m(t)$ constitutes the best (in the mean-square sense) estimate (filter) of the quantity $x(t)$ constructed on the basis of the measured realization of the process $y(s)$, $0 \leqslant s \leqslant t$, and $D(t)$ (by virtue of the assumptions of Sect. 1) coincides with the unconditional dispersion of the difference $x(t) - m(t)$.

Let us introduce the fundamental matrix $z(t, s)$ of system (1.1) by means of the relations

$$dz(t, s)/dt = A(t)z(t, s), \qquad z(s, s) = I \tag{2.1}$$

where I is an identity matrix. Before stating the optimal problems under investigation exactly, we propose to show that the function $m(t)$ is a solution of the system of stochastic differential equations

$$dm(t) = D(t)z'(t-h, t)Q'(t)B^{-1}(t)[dy(t) - Q(t)z(t-h, t)m(t)dt +$$

$$+Q(t)\int_{t-h}^{t} z(t-h, s)f(s)\,ds\,dt] + (A(t)m(t) + f(t))dt, \qquad m(0) = m_0 \tag{2.2}$$

and that the matrix $D(t)$ is defined by the equations

$$D^{\cdot}(t) = -D(t)z'(t-h,\ t)Q'(t)B^{-1}(t)Q(t)z(t-h,\ t)D(t) + \\ + A(t)D(t) + D(t)A'(t), \qquad D(0) = D_0 \tag{2.3}$$

which constitute a system of ordinary differential equations.

Formulas (2.2) and (2.3) for $h = 0$ were obtained in well-known paper [4].

Theorem 2.1. Let the assumptions of Sect. 1 hold. Then the functions $m(t)$ and $D(t)$ satisfy Eqs. (2.2), (2.3).

Proof. By virtue of formula (1.1) the function $\varphi(t) = x(t-h)$ is a solution of the equation

$$\varphi^{\cdot}(t) = A(t-h)\varphi(t) + f(t-h) \tag{2.4}$$

with the initial condition

$$\varphi(0) = z(-h,\ 0)\,x(0) + \int_0^{-h} z(-h,\ s)\,f(s)\,ds \tag{2.5}$$

Let us denote by $m_-(t)$ the best mean-square estimate of $\varphi(t)$ based on the tracking data $y(s),\ 0 \leqslant s \leqslant t$ and let $D_-(t)$ denote the dispersion of the difference $x(t) - m_-(t)$.

By virtue of (2.4), (2.5) and the results of [4], the function $m_-(t)$ satisfies the following equations:

$$dm_-(t) = D_-(t)Q'(t)B^{-1}(t)(dy(t) - Q(t)m_-(t)dt) + (A(t-h)m_-(t) + \\ + f(t-h))\,dt, \qquad m_-(0) = M\varphi(0) \tag{2.6}$$

and the matrix $D_-(t)$ is the solution of the system of ordinary differential equations

$$D^{\cdot}_-(t) = A(t-h)D_-(t) + D_-(t)A'(t-h) - D_-(t)Q'(t)B^{-1}(t)Q(t)D_-(t) \tag{2.7}$$

defined by the initial conditions

$$D_-(0) = M(\varphi(0) - m_-(0))^2 \tag{2.8}$$

Equation (1.1) is determinate, so that

$$D(t) = z(t,\ t-h)\ D_-(t)\,z'(t,\ t-h) \tag{2.9}$$

With allowance for (2.1) and the associated equation for the fundamental matrix $z(t,\ s)$, namely

$$\partial z(t,\ s)/\partial s = -z(t,\ s)A(s), \qquad z(s,\ s) = I$$

we find that the total derivative $z^{\cdot}(t,\ t-h)$ is

$$z^{\cdot}(t,t-h) = A(t)z(t,t-h) - z(t,\ t-h)A(t-h) \tag{2.10}$$

But by virtue of (2.9) we have

$$D^{\cdot}(t) = z^{\cdot}(t,\ t-h)D_-(t)z'(t,\ t-h) + z(t,\ t-h)D_-^{\cdot}(t)z'(t,\ t-h) + \\ + z(t,\ t-h)D_-(t)z^{\cdot\prime}(t,\ t-h) \tag{2.11}$$

Let us transform expression (2.11) with the aid of formulas (2.7), (2.9). First of all, we infer from (2.9), (2.10) that

$$z^{\cdot}(t,\ t-h)D_-(t)z'(t,\ t-h) = A(t)D(t) - \\ - z(t,t-h)A(t-h)D_-(t)z'(t,\ t-h) \tag{2.12}$$

and similarly that

$$z(t,\ t-h)D_-(t)z^{\cdot\prime}(t,\ t-h) = D(t)A'(t) - \\ - z(t,\ t-h)D_-(t)A'(t-h)z'(t,\ t-h) \tag{2.13}$$

Since

$$z(t,\ t_1)\,z(t_1,\ s) = z(t,\ s) \tag{2.14}$$

it follows by (2.7) that

$$z(t, t-h) D_-^{\cdot}(t) z'(t, t-h) = z(t, t-h)[A(t-h) D_-(t) + D_-(t) A'(t-h)] z'(t, t-h) - D(t) z'(t-h, t) Q'(t) B^{-1}(t) Q(t) z(t-h, t) D(t) \qquad (2.15)$$

Thus, making use of formulas (2.8), (2.9) and substituting (2.12), (2.13), (2.15) into (2.11), we see that relations (2.3) are indeed valid.

To prove Eq. (2.2) we note that by virtue of (1.1)

$$m(t) = z(t, t-h) m_-(t) + \int_{t-h}^{t} z(t, s) f(s)\, ds \qquad (2.16)$$

Let us compute the stochastic differential of the function $m(t)$. From (2.1), (2.6), (2.10) and (2.16) we infer that

$$dm(t) = A(t) z(t, t-h) m_-(t)\, dt + A(t) \int_{t-h}^{t} z(t, s) f(s)\, ds\, dt +$$

$$+ z(t, t-h) D_-(t) Q'(t) B^{-1}(t)[dy(t) - Q(t) m_-(t)\, dt] + f(t)\, dt \qquad (2.17)$$

Let us transform the individual terms in the right side of (2.17). By virtue of (2.16) we have

$$A(t) z(t, t-h) m_-(t) = A(t) m(t) - A(t) \int_{t-h}^{t} z(t, s) f(s)\, ds \qquad (2.18)$$

Further, by virtue of (2.9), (2.14) we have

$$z(t, t-h) D_-(t) Q'(t) B^{-1}(t) = D(t) z'(t-h, t) Q'(t) B^{-1}(t) \qquad (2.19)$$

Moreover, recalling (2.9), (2.14), (2.16) we find that

$$Q(t) m_-(t) = Q(t) z(t-h, t) m(t) - Q(t) \int_{t-h}^{t} z(t-h, s) f(s)\, ds \qquad (2.20)$$

Hence, the substitution of (2.18)–(2.20) into (2.17) proves Eqs. (2.2). Theorem 2.1 has been proved.

Let us denote by $V(t)$ the nonnegative-definite symmetric matrix

$$V(t) = z'(t-h, t) Q'(t) B^{-1}(t) Q(t) z(t-h, t) \qquad (2.21)$$

and rewrite Eq. (2.3), which constitutes a Bernoulli matrix equation, in the form

$$D^{\cdot}(t) = A(t) D(t) + D(t) A'(t) - D(t) V(t) D(t), \qquad D(t_0) = D_0 \qquad (2.22)$$

It is easy to show that if the requirements of Sect. 1 are fulfilled, then for any nonnegative-definite matrix $V(t)$ there exist a matrix $Q(t)$ and a positive-definite matrix $B(t)$ which satisfy Eq. (2.21). Repeating the arguments of [5] (p. 420), we find that the determinant $\det D(t)$ of the matrix $D(t)$ satisfies the Jacobi identity

$$\det D(t) = \det D_0 \exp\left\{\int_0^t Sp\,[A(s) - V(s) D(s) + A'(s)]\, ds\right\}$$

(here $SpA(t)$ is the trace of the matrix $A(t)$), whence it follows that the matrix $D(t)$ is nonsingular for all values of the argument t. Hence, there exists a matrix $\alpha(t) = D^{-1}(t)$ defined by the following conditions by virtue of (2.22):

$$\alpha^{\cdot}(t) = -A'(t)\alpha(t) - \alpha(t) A(t) + V(t), \qquad \alpha(0) = \alpha_0 = D_0^{-1} \qquad (2.23)$$

Occasionally, in order to emphasize the dependence of the solution $\alpha(t)$ of Eq. (2.23) on the initial conditions and on the function $V(t)$, we denote it by the symbol $\alpha(\alpha_0, V, t)$.

3. Problem 1. Suppose that we are given the positive-definite matrices $\alpha_0 = D_0^-$, $\alpha_T = D_T^{-1}$ such that the matrix

$$R = \alpha_T - [z'(T, 0)]^{-1}\alpha_0 z(T, 0)^{-1}$$

is nonnegative-definite. We are to find a nonnegative-definite matrix $V_0(t)$ which for each $i = 1, \ldots, l$ minimizes the expression

$$\int_0^T \sum_{j=1}^{l} ([z'(T,s)]^{-1} V_0(s))_{ij}^2 ds$$

(here and below $(A)_{ij}$ denotes the ijth element of the matrix A) for which $\alpha(\alpha_0, V_0, T) = \alpha_T$.

If $V(t) \equiv 0$, then by virtue of (2.22), (2.23) the function

$$z(T, 0)\alpha_0^{-1}z'(T, 0) = z(T, 0)\, D_0 z'(T, 0)$$

is the dispersion matrix of the unknown vector $x(t)$ at the instant $t = T$ in the absence of observations. As a result of measurements, probabilistic judgements about the quantity $x(T)$ must undergo refinement in some sense, and this is reflected in the requirement of nonnegative definiteness of the matrix R; this requirement implies, in particular, that the conditional dispersion of the quantities $x_i(T)$ during tracking is not larger than in the absence of observations. The validity of the latter statement follows from the nonnegative definiteness of the matrix

$$z(T, 0)\alpha_0^{-1}z'(T, 0)R\alpha_T^{-1} = z(T, 0)D_0 z'(T, 0) - D_T$$

Theorem 3.1. Let the coefficients of Eq. (1.1) satisfy the requirements of Sect. 1. Then the function

$$V(t) = z'(T, t)RG^{-1}z'(T, t)^{-1}, \qquad 0 \leqslant t \leqslant T$$

where

$$G = \int_0^T z'(T,s)^{-1} z(T,s)^{-1} ds$$

yields the solution of Problem 1.

Proof. The solution of Eq. (2.23) can be written as (see [1], p. 100)

$$\alpha(t) = z'(t, 0)^{-1}\alpha_0 z(t, 0)^{-1} + \int_0^t z'(t, s)^{-1}V(s)\, z(t, s)^{-1} ds$$

Setting $t = T$ in this equation, we find that if Problem 1 had a solution, the following formula would be valid:

$$R = \int_0^T \omega(s)\, z(T, s)^{-1} ds \tag{3.1}$$

where the matrix $\omega(s) = z'(T, s)^{-1} V(s)$. Hence, to solve Problem 1 we need merely find a matrix $\omega(s)$ which for every $i = 1, \ldots, l$ minimizes the expression

$$\int_0^T \sum_{j=1}^{l} (\omega(s))_{ij}^2 ds \tag{3.2}$$

and satisfies Eq. (3.1). Now let us expand the class of matrices ω among which we seek the solution of Problem 1 to the set of matrices with square-integrable elements. Since by (3.1) we have

$$(R)_{ij} = \int_0^T \sum_{k=1}^{l} (\omega(s))_{ik} \, (z(T, s)^{-1})_{kj} \, ds \tag{3.3}$$

it follows that the ancillary optimal problem can be stated as follows: we are to find l row vectors having square-integrable elements $(\omega(s))_{ik}$ (the integer i, $1 \leqslant i \leqslant l$, is the number of the vector; the number k runs through the values from 1 to l for each i) which minimize relations (3.2) and satisfy relations (3.3).

Let i, $1 \leqslant i \leqslant l$, be an arbitrary fixed number. Let us consider Eqs. (3.3) for this value of i. We denote by β the row vector with the coordinates

$$\beta_j = (R)_{ij}, \qquad 1 \leqslant j \leqslant l \tag{3.4}$$

and by $\varphi_0'(s)$ the function equal to the ith row vector which yields the solution of the ancillary problem. Computing the function $\varphi_0(s)$ by the method of definite Lagrange multipliers (for example, see [6], Sect. 18), we find with allowance for (3.2) and (3.4) that

$$\varphi_0'(s) = \beta G^{-1} z'(T, s)^{-1} \tag{3.5}$$

We recall that formula (3.5) was derived for an arbitrary but fixed value of the index i. Hence, since i is arbitrary, we finally infer from (3.5) that the solution $\omega(s)$ of the ancillary optimal problem is given by the formula

$$\omega(s) = RG^{-1} z'(T, s)^{-1}, \quad 0 \leqslant s \leqslant T$$

This and the nonnegative definiteness of the matrix

$$V(t) = z'(T, s) RG^{-1} z(T, s) [z(T, s)^{-1} z'(T, s)^{-1}]$$

imply that the matrix $V(t)$ is the solution of Problem 1. Theorem 3.1 has been proved.

Example 3.1. Let the unknown two-dimensional vector x_0 have an a priory Gaussian distribution with the parameters m_0, D_0, and let the observer have available to him the quantity

$$y(t) = \int_0^t Q(s) \, ds x_0 + \xi(t), \quad 0 \leqslant t \leqslant T$$

where $\xi(t)$ is a two-dimensional Wiener process. We are to choose $Q(t)$ in such a way that the dispersion of the difference between the unknown vector x_0 and the estimate $m(t)$ of this vector based on the tracking data is equal for $t = T$ to a given matrix D_T such that $D_0 - D_T$ is nonnegative-definite. Setting

$$x^{\cdot}(t) = 0, \; x(0) = x_0, \qquad y(t) = \int_0^T Q(s) \, x(s) \, ds + \xi(t)$$

for $0 \leqslant t \leqslant T$ and applying Theorem 3.1 (in whose conditions $A(t)$ is a zero matrix, $z(T, t)$ is equal to the identity matrix I, $G = IT$ and $R = D_T^{-1} - D_0^{-1}$), we find that the matrix $Q(t)$ is constant and with allowance for formula (2.21) satisfies the equation $Q'Q = RT^{-1}$, the existence of whose solution follows from the nonnegative definiteness of the matrix R.

4. Pontriagin's maximum principle [7] enables us to consider other variants of the optimal tracking problem and to reduce their solution to a system of transcendental

equations. Since the exact statement of other optimal tracking problems and their reduction to a system of transcendental equations for $h > 0$ is in no way different from the statement of the problem for $h = 0$ studied in [1], we shall merely illustrate the foregoing results by means of examples.

Example 4.1. Suppose we are given a one-dimensional equation of the form (1.1) with a constant coefficient a,

$$x^{\cdot}(t) = ax(t) + f(t) \tag{4.1}$$

under the initial condition $x(0) = x_0$. The random quantity $x(0)$ is Gaussian with a known dispersion $D_0 > 0$. The quantity

$$\int_0^t b(s)\, x(s-h)\, ds + \sigma\xi(t)$$

is observed in the interval $[0, T]$. Here the constant $\sigma \neq 0$ and $\xi(t)$ is a Wiener process which does not depend on $x(0)$. By suitable choice of the function $b(t)$ which is piecewise-continuous from the left, is equal either to zero or to a constant $b \neq 0$ for any t, and satisfies the condition

$$\int_0^T b(s)\, ds = bT_0, \qquad T_0 < T \tag{4.2}$$

we are required to minimize the expression

$$J = \beta_1 D(T) + \beta \int_0^T D(s)\, ds$$

where the nonnegative constants β and β_1 are such that $\beta + \beta_1 > 0$. Requirement (4.2) means, by virtue of the definition of the function $b(t)$, that the sum duration of the tracking process is given.

This example was previously studied in [1] for $h = 0$, $\beta = 0$, $\beta_1 = 1$.

Equation (2.23) with allowance for (2.21) becomes

$$\alpha^{\cdot}(t) = -2a\alpha(t) + V(t), \qquad 0 < t \leqslant T$$
$$\alpha(0) = D_0^{-1}, \qquad V(t) = \sigma^{-2} b(t) \exp(-2ah)$$

Making use of the maximum principle ([7], pp. 75-79), we can readily show that

$$b_0(t) = b \quad \text{if} \quad \psi(t) + c > 0 \tag{4.3}$$
$$b_0(t) = 0 \quad \text{if} \quad \psi(t) + c < 0$$

where the constant c is chosen in such a way as to satisfy requirement (4.2), and where the associated variable $\psi(t)$ is defined by the relations

$$\psi^{\cdot}(t) = 2a\psi(t) - \beta\alpha^{-2}(t), \qquad 0 < t \leqslant T$$
$$\psi(T) = \beta_1 \alpha^{-2}(T) \tag{4.4}$$

These equations imply that the function $\psi(t)$ decreases monotonically for $a < 0$. Hence, for $a < 0$ we have

$$b_0(t) = b \quad \text{if} \quad 0 \leqslant t \leqslant T - T_0$$
$$b_0(t) = 0 \quad \text{if} \quad T - T_0 < t \leqslant T$$

The optimal tracking law is of the same form for $a = 0$, $\beta > 0$. Similarly, for $a > 0$, $\beta = 0$ we have (by virtue of (4.4))

$$b_0(t) = 0, \ 0 \leqslant t \leqslant T_0 - T; \ b_0(t) = b, \ T_0 - T < t \leqslant T$$

If $a = 0, \beta = 0$, then the value of the functional J does not depend on the tracking law. Finally, let us show that if $a > 0, \beta > 0$, then there exists just one tracking interval, i.e. that there exists an instant $t_1 \leqslant T - T_0$ such that $b_0(t) = b$ for

$$t \in (t_1, t_1 + T_0] \text{ and } b_0(t) = 0 \text{ for } t \overline{\in} (t_1, t_1 + T_0]$$

To prove this by reduction ad absurdum, we assume that there exist several nonadjacent intervals $(t_1, s_i]$ where the function $b_0(t)$ defined by formulas (4.3) is different from zero. Let us investigate the behavior of the associated variable $\psi(t)$ for $s_i \leqslant t \leqslant t_{i+1}$. Let us set $r(t) = \psi^{\cdot}(t)$. By virtue of our assumption

$$r(s_i) \leqslant 0 \tag{4.5}$$

Moreover, by virtue of (4.4) we have

$$r^{\cdot}(t) = 2ar(t) + 2\beta\alpha^{\cdot}(t)\alpha^{-2}(t)$$

This and (4.5), together with the fact that $\alpha^{\cdot}(t) < 0$ for $s_i < t \leqslant t_{i+1}$, we infer that $\psi(t)$ decreases monotonically in the interval $[s_i, t_{i+1}]$. However, this contradicts the above assumption whereby $b_0(t) = b$ for $t_{i+j} < t \leqslant s_{i+j}$, $j = 0, 1$. Thus, for $\beta > 0$, $a > 0$ the problem of finding the optimal tracking law reduces to the problem of finding the minimum of a scalar function of a single variable t_1. To this end we must find $\alpha(t)$ for $V(t) = b_0(t)\sigma^{-2}$ and substitute the resulting function $\alpha(t)$ into the functional J, which thereby becomes a scalar function of the variable t_1. Specifically, it is easy to show that for the values $\beta = 0, \beta_1 = 1, \sigma = 1$ we have

$$D(T) = e^{2aT}[D_0^{-1} + e^{-2ah} b^{-1} \frac{1}{2a} (e^{2aT} - e^{2a(T-T_0)})]^{-1}, \qquad a > 0$$

$$D(T) = e^{2aT}[D_0^{-1} + e^{-2ah} (2ab)^{-1} (e^{2aT_0} - 1)]^{-1}, \qquad a < 0$$

For $a = 0$, the value of $D(T)$ which does not depend on the tracking law is given by

$$D(T) = [D_0^{-1} + b^{-1} T_0]^{-1}, \qquad a = 0$$

Example 4.2. In addition to the case of continuous tracking analyzed above, it ia also possible to make observations at discrete instants. The latter entails optimal selection of the instants of measurement. Turning once again to Eq. (4.1), let us suppose that at discrete instants $t_1, \ldots, t_r$ in the interval $[0, T]$ we observe the quantity

$$x(t - h) + \zeta(t), \qquad h \geqslant 0 \tag{4.6}$$

where $\zeta(t)$ is a white noise of constant intensity $b > 0$.

We are required to choose the numbers t_i in such a way as to minimize the conditional dispersion $Dx(T)$. On the basis of (2.23) we readily infer that $Dx(T)$ in this case is equal to

$$e^{2aT}\left[D_0^{-1} + e^{-2ah} b^{-1} \sum_{i=1}^{r} e^{2at_i}\right]^{-1} \tag{4.7}$$

This implies that for $a > 0$ we must set all $t_j = T$, and that for $a < 0$ the numbers $t_i = 0, \ i = 1, \ldots, r$.

Examples 4.1 and 4.2 indicate that with a noise of constant intensity the presence of a lag in the tracking channel can increase the conditional dispersion for $a > 0$, i.e. it can result in a deterioration of data on the position of the unknown object relative to the case $h = 0$. On the other hand, if $a < 0$, the presence of lag can have the reverse effect of decreasing the conditional dispersion.

Note. In stating the above tracking problem we assumed that the a priori distribution of the solution $x(t)$ of Eq. (1.1) is known for $t = 0$, and that data obtained during tracking in the interval $[0, T]$ can be used to determine the prehistory of $x(t)$ for $-h \leqslant t \leqslant 0$ (by virtue of (1.2)). The assumption is justified if the motion of the unknown object is defined by Eq. (1.1) for all t. If we also know that the motion of this object is described by Eq. (1.1) beginning at $t = 0$ only, then we must alter the statement of the tracking problem as follows: The a priori distribution of the function $x(t)$ is given for $t = 0$ as before, but tracking is conducted at times $t \geqslant h$. Since Eq. (1.1) is determinate, this problem is readily reducible to that considered above simply by taking $t_0 = h$ as the initial point and computing the a priori distribution of $x(h)$ with the aid of (1.1). Here the tracking time decreases by the amount h in comparison with the case $h = 0$, however. Moreover, there is no gain in data due to knowledge of the prehistory of the process $x(t)$, which accounts for the decrease in dispersion for $a < 0$ in Examples 4.1 and 4.2. It is not difficult to show that if in Example 4.2 quantity (4.6) is observed at discrete instants t_i, $h \leqslant t_i \leqslant T$, then for $a > 0$ we must set all $t_i = T$, and for $a < 0$ all $t_i = h$. In other words, by virtue of (4.7) the conditional dispersion increases for $a > 0$; for $a < 0$ it remains the same as in the case $h = 0$.

The author is grateful to F. L. Chernous'ko for his valuable comments on the above results.

BIBLIOGRAPHY

1. Chernous'ko, F. L., On optimization of the tracking process. PMM Vol. 33, №1, 1969.
2. Solianik, A. I. and Chernous'ko, F. L., Optimizing the tracking process under random perturbations. PMM Vol. 33, №4, 1969.
3. Iashin, A. I., On the selection of the optimal tracking process. Izv. Akad. Nauk SSSR, Tekhnicheskaia Kibernetika №2, 1969.
4. Kalman, R. E. and Bucy, R. C., New results in linear filtering and prediction theory. Trans. ASME, ser. D., J. Basic Engng. Vol. 83, №1, 1961.
5. Gantmakher, F. R., Theory of Matrices, 2nd ed. Moscow, "Nauka", 1968.
6. Krasovskii, N. N., Theory of Motion Control. Moscow, "Nauka", 1968.
7. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Optimal Processes, 2nd ed., Moscow, "Nauka", 1969.

Translated by A. Y.